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# Variational formulations for TFEM of piezoelectricity

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**Abstract**

The paper presents a family of variational formulations of Trefftz finite elements wherein the assumed internal displacement and electric potential fields a priori fulfil the governing differential equations of the problem over the element sub-domain, while the inter-element continuity and the boundary conditions are enforced using a modified variational principle together with an independent frame field defined on each element boundary. It is based on four free energy densities, each with two kinds of independent variables as basic independent variables, i.e. $(\boldsymbol{\sigma}, \mathbf{D})$, $(\boldsymbol{\varepsilon}, \mathbf{E})$, $(\boldsymbol{\varepsilon}, \mathbf{D})$, and $(\boldsymbol{\sigma}, \mathbf{E})$. Based on the assumed intra-element and frame fields, an element stiffness matrix equation is obtained which is implemented into computer programs for numerical analysis. Some numerical examples are considered to show the application of the proposed formulation.
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*Keywords:* Trefftz method; Piezoelectric; Variational principle; Finite element

## 1. Introduction

Variational functionals are essential and play a central role in the formulation of the fundamental governing equations in the Trefftz finite element method (TFEM). They are at the heart of many numerical methods such as boundary element methods, finite volume methods and TFEMs (Qin, 2000). During the past decades, much work has been done concerning variational formulations for Trefftz numerical methods (Jirousek, 1993; Jirousek and Zielinski, 1993; Piltner, 1985; Qin, 2000). Piltner (1985) presented two different variational formulations to treat special elements with holes or cracks. The formulation consists of a conventional potential energy and a least square functional. The least square functional is not added as a penalty function to the potential functional, but is minimized separately for the special elements considered. Jirousek (1993) developed a variational functional in which either the displacement conformity or the reciprocity of the conjugate tractions is enforced at the element interface. Jirousek and Zielinski (1993) obtained two complementary hybrid Trefftz formulations based on a weighted residual method. The dual

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formulations enforced more strongly the reciprocity of boundary tractions than the conformity of the displacement fields. Qin (2000) presented a modified variational principle based hybrid Trefftz displacement frame. The variational functional of Qin (2000) is, however, limited in that nodes containing unknown displacements must connect with at least one inter-element boundary. To remove this limitation, we present a set of variational functional for TFEM and apply it to piezoelectric problems in this paper. It entails four variational functionals which are based on four free energy densities, each with two kinds of independent variables as basic independent variables, i.e. $(\boldsymbol{\sigma}, \mathbf{D})$, $(\boldsymbol{\varepsilon}, \mathbf{E})$, $(\boldsymbol{\varepsilon}, \mathbf{D})$, and $(\boldsymbol{\sigma}, \mathbf{E})$. The stationary conditions of these variational functionals and their theorem on the existence of extremum are also discussed. The stationary conditions are displacement and electric potential conditions on the boundary, surface traction and surface charge condition, and inter-element continuity condition. These functionals are suitable for piezoelectric analysis with TFEM. Numerical results are found to agree well with the analytical solutions.

## 2. Basic equations for TFEM of piezoelectricity

### 2.1. Basic field equations and boundary conditions

Consider a linear piezoelectric material, in which the differential governing equations in the Cartesian coordinates $x_i$ $(i = 1, 2, 3)$ are given by

$$\sigma_{ij,j} + \bar{b}_i = 0, \quad D_{i,i} + \bar{q}_b = 0 \quad \text{in } \Omega \tag{1}$$

where $\sigma_{ij}$ is the stress tensor, $D_i$ is the electric displacement vector, a comma denotes partial differentiation with respect to the coordinate $x_i$, $\bar{b}_i$ is the body force vector, $\bar{q}_b$ is the electric charge density, $\Omega$ is the solution domain, and the Einstein summation convention over repeated indices is used. For an anisotropic piezoelectric material, the constitutive relation is

$$\varepsilon_{ij} = -\frac{\partial H(\boldsymbol{\sigma}, \mathbf{D})}{\partial \sigma_{ij}} = s^D_{ijkl}\sigma_{kl} + g_{kij}D_k, \quad E_i = \frac{\partial H(\boldsymbol{\sigma}, \mathbf{D})}{\partial D_i} = -g_{ikl}\sigma_{kl} + \lambda^{\sigma}_{ik}D_k \tag{2a}$$

for $(\boldsymbol{\sigma}, \mathbf{D})$ as basic variables,

$$\sigma_{ij} = \frac{\partial H(\boldsymbol{\varepsilon}, \mathbf{E})}{\partial \varepsilon_{ij}} = c^E_{ijkl}\varepsilon_{kl} - e_{kij}E_k, \quad D_i = -\frac{\partial H(\boldsymbol{\varepsilon}, \mathbf{E})}{\partial E_i} = e_{ikl}\varepsilon_{kl} + \kappa^{\varepsilon}_{ik}E_k \tag{2b}$$

for $(\boldsymbol{\varepsilon}, \mathbf{E})$ as basic variables,

$$\sigma_{ij} = \frac{\partial H(\boldsymbol{\varepsilon}, \mathbf{D})}{\partial \sigma_{ij}} = c^D_{ijkl}\varepsilon_{kl} + h_{kij}D_k, \quad E_i = \frac{\partial H(\boldsymbol{\varepsilon}, \mathbf{D})}{\partial D_i} = h_{ikl}\varepsilon_{kl} + \lambda^{\varepsilon}_{ik}D_k \tag{2c}$$

for $(\boldsymbol{\varepsilon}, \mathbf{D})$ as basic variables, and

$$\varepsilon_{ij} = -\frac{\partial H(\boldsymbol{\sigma}, \mathbf{E})}{\partial \sigma_{ij}} = s^E_{ijkl}\sigma_{kl} + d_{kij}D_k, \quad D_i = -\frac{\partial H(\boldsymbol{\sigma}, \mathbf{E})}{\partial E_i} = d_{ikl}\sigma_{kl} + \kappa^{\sigma}_{ik}E_k \tag{2d}$$

for $(\boldsymbol{\sigma}, \mathbf{E})$ as basic variables, with

$$H(\boldsymbol{\sigma}, \mathbf{D}) = -\tfrac{1}{2}s^D_{ijkl}\sigma_{ij}\sigma_{kl} + \tfrac{1}{2}\lambda^{\sigma}_{ij}D_iD_j - g_{kij}\sigma_{ij}D_k \tag{3a}$$

$$H(\boldsymbol{\varepsilon}, \mathbf{E}) = \tfrac{1}{2}c^E_{ijkl}\varepsilon_{ij}\varepsilon_{kl} - \tfrac{1}{2}\kappa^{\varepsilon}_{ij}E_iE_j - e_{kij}\varepsilon_{ij}E_k \tag{3b}$$

$$H(\boldsymbol{\varepsilon}, \mathbf{D}) = \tfrac{1}{2}c^D_{ijkl}\varepsilon_{ij}\varepsilon_{kl} + \tfrac{1}{2}\lambda^{\varepsilon}_{ij}D_iD_j + h_{kij}\varepsilon_{ij}D_k \tag{3c}$$

$$H(\boldsymbol{\sigma}, E) = -\tfrac{1}{2}s^{E}_{ijkl}\sigma_{ij}\sigma_{kl} - \tfrac{1}{2}\kappa^{\sigma}_{ij}E_iE_j - d_{kij}\sigma_{ij}E_k \tag{3d}$$

where $c^{E}_{ijkl}$, $c^{D}_{ijkl}$ and $s^{E}_{ijkl}$, $s^{D}_{ijkl}$ are the stiffness and compliance coefficient tensor for $\mathbf{E} = 0$ or $\mathbf{D} = 0$, $\kappa^{\sigma}_{ij}$, $\kappa^{\varepsilon}_{ij}$ and $\lambda^{\sigma}_{ij}$, $\lambda^{\varepsilon}_{ij}$ are the permittivity matrix and the conversion of the permittivity constant matrix for $\boldsymbol{\sigma} = 0$ or $\boldsymbol{\varepsilon} = 0$, $\varepsilon_{ij}$ and $E_i$ are, respectively, the elastic strain tensor and the electric field intensity vector, $e_{kij}$ is piezoelectric stress constants, $g_{kij}$ is piezoelectric strain constants. These constants have the following relations:

$$\begin{aligned}&\mathbf{c}^E = (\mathbf{s}^E)^{-1}, \quad e = \mathbf{c}^E\mathbf{d}, \quad \boldsymbol{\kappa}^{\varepsilon} = \boldsymbol{\kappa}^{\sigma} - \mathbf{d}^{\mathrm{T}}\mathbf{c}^e\mathbf{d}, \quad \mathbf{s}^D = (\mathbf{c}^E)^{-1} - \mathbf{d}(\boldsymbol{\kappa}^{\sigma})^{-1}\mathbf{d}^{\mathrm{T}}, \quad \mathbf{g} = \mathbf{d}(\boldsymbol{\kappa}^{\sigma})^{-1},\\ &\mathbf{c}^D = \mathbf{c}^E - \mathbf{c}^E\mathbf{d}(\boldsymbol{\kappa}^{\varepsilon})^{-1}\mathbf{d}^{\mathrm{T}}\mathbf{c}^E, \quad \mathbf{h} = -\mathbf{c}^E\mathbf{d}(\boldsymbol{\kappa}^{\varepsilon})^{-1}, \quad \boldsymbol{\lambda}^{\varepsilon} = (\boldsymbol{\kappa}^{\varepsilon})^{-1}, \quad \boldsymbol{\lambda}^{\sigma} = (\boldsymbol{\kappa}^{\sigma})^{-1}\end{aligned} \tag{4}$$

where superscript '$T$' denotes the transposition of a matrix.

The relation between the strain tensor and the displacement, $u_i$, is given by

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \tag{5}$$

and the electric field components are related to the electric potential $\phi$ by

$$E_i = -\phi_{,i} \tag{6}$$

The boundary conditions of the boundary value problem (1)–(6), can be given by:

$$u_i = \bar{u}_i \quad \text{on } \Gamma_u \tag{7}$$

$$t_i = \sigma_{ij}n_j = \bar{t}_i \quad \text{on } \Gamma_t \tag{8}$$

$$D_n = D_in_i = -\bar{q}_n = \overline{D}_n \quad \text{on } \Gamma_D \tag{9}$$

$$\phi = \bar{\phi} \quad \text{on } \Gamma_{\phi} \tag{10}$$

where $\bar{u}_i$, $\bar{t}_i$, $\bar{q}_n$ and $\bar{\phi}$ are, respectively, prescribed boundary displacement, traction vector, surface charge and electric potential, an overhead bar denotes prescribed value, $\Gamma = \Gamma_u + \Gamma_t = \Gamma_D + \Gamma_{\phi}$ is the boundary of the solution domain $\Omega$.

Moreover, in the Trefftz finite element form, Eqs. (1)–(10) should be completed by the following inter-element continuity requirements:

$$u_{ie} = u_{if}, \quad \phi_e = \phi_f \quad (\text{on } \Gamma_e \cap \Gamma_f, \text{ conformity}) \tag{11}$$

$$t_{ie} + t_{if} = 0, \quad D_{ne} + D_{nf} = 0 \quad (\text{on } \Gamma_e \cap \Gamma_f, \text{ reciprocity}) \tag{12}$$

where '$e$' and '$f$' stand for any two neighboring elements. Eqs. (1)–(12) are taken as the basis to establish the modified variational principle for Trefftz finite element analysis of piezoelectric materials.

### 2.2. Assumed fields

The main idea of the TFEM is to establish a finite element (FE) formulation whereby the intra-element continuity is enforced on a non-conforming internal displacement field chosen so as to a priori satisfy the governing differential equation of the problem under consideration (Qin, 2000). In other words, as an obvious alternative to the Rayleigh–Ritz method as a basis for a FE formulation, the model here is based on the method of Trefftz (1926). With this method the solution domain is subdivided into elements, and over each element "$e$," the assumed intra-element fields are

$$\mathbf{u} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ \phi \end{Bmatrix} = \begin{Bmatrix} \breve{u}_1 \\ \breve{u}_2 \\ \breve{u}_3 \\ \breve{\phi} \end{Bmatrix} + \begin{Bmatrix} \mathbf{N}_1 \\ \mathbf{N}_2 \\ \mathbf{N}_3 \\ \mathbf{N}_4 \end{Bmatrix} \mathbf{c} = \breve{\mathbf{u}} + \sum_{j=1} \mathbf{N}_j \mathbf{c}_j = \breve{\mathbf{u}} + \mathbf{Nc} \tag{13}$$

where $\mathbf{c}_i$ stands for undetermined coefficient, and $\breve{\mathbf{u}}(= \{\breve{u}_1, \breve{u}_2, \breve{u}_3, \breve{\phi}\}^{\mathrm{T}})$ and $\mathbf{N}$ are known functions. If the governing differential equation (1) is rewritten in a general form

$$\Re \mathbf{u}(\mathbf{x}) + \bar{\mathbf{b}}(\mathbf{x}) = 0 \quad (\mathbf{x} \in \Omega_e) \tag{14}$$

where $\Re$ stands for the differential operator matrix for Eq. (1), $\mathbf{x}$ for the position vector, $\bar{\mathbf{b}}(= \{\{\bar{b}_1, \bar{b}_2, \bar{b}_3, \bar{q}_b\}^{\mathrm{T}})$ for the known right-hand side term, the overhead bar indicates the imposed quantities and $\Omega_e$ stands for the $e$th element sub-domain, then $\breve{\mathbf{u}} = \breve{\mathbf{u}}(\mathbf{x})$ and $\mathbf{N} = \mathbf{N}(\mathbf{x})$ in Eq. (13) have to be chosen such that

$$\Re \breve{\mathbf{u}} + \bar{\mathbf{b}} = 0 \quad \text{and} \quad \Re \mathbf{N} = 0 \tag{15}$$

everywhere in $\Omega_e$. A complete system of homogeneous solutions $\mathbf{N}_j$ can be generated by way of the solution in Stroh formalism

$$\mathbf{u} = 2\,\mathrm{Re}\{\mathbf{A}\langle f(z_\alpha)\rangle \mathbf{c}\} \tag{16}$$

where 'Re' stands for the real part of a complex number, $\mathbf{A}$ is the material eigenvector matrix which was well defined in the reference (Qin, 2001, pp. 17, 18), $\langle f(z_\alpha)\rangle = \mathrm{diag}[f(z_1)\ f(z_2)\ f(z_3)\ f(z_4)]$ is a diagonal $4\times4$ matrix, while $f(z_i)$ is an arbitrary function with argument $z_i = x_1 + p_i x_2 \cdot p_i$ $(i = 1\text{–}4)$ are the material eigenvalues. Of particular interest is a complete set of polynomial solutions which may be generated by setting in Eq. (16) in turn

$$\left.\begin{matrix} f(z_\alpha) = z_\alpha^k \\ f(z_\alpha) = \mathrm{i} z_\alpha^k \end{matrix}\right\} \quad (k = 1, 2, \ldots) \tag{17}$$

where $\mathrm{i} = \sqrt{-1}$. This leads, for $\mathbf{N}_j$ of Eq. (13), to the following sequence

$$\mathbf{N}_{2j} = 2\,\mathrm{Re}\{\mathbf{A}\langle z_\alpha^j\rangle\} \tag{18}$$

$$\mathbf{N}_{2j+1} = 2\,\mathrm{Re}\{\mathbf{A}\langle \mathrm{i} z_\alpha^j\rangle\} \tag{19}$$

The unknown coefficient $\mathbf{c}$ may be calculated from the conditions on the external boundary and/or the continuity conditions on the inter-element boundary. Thus various Trefftz element models can be obtained by using different approaches to enforce these conditions. In the majority of cases a hybrid technique is used, whereby the elements are linked through an auxiliary conforming displacement frame which has the same form as in the conventional FE method. This means that, in the Trefftz finite element approach, a conforming electric potential and displacement (EPD) field should be independently defined on the element boundary to enforce the field continuity between elements and also to link the coefficient $\mathbf{c}$, appearing in Eq. (13), with nodal EPD $\mathbf{d}$ $(= \{\mathbf{d}\})$. The frame is defined as

$$\tilde{\mathbf{u}}(\mathbf{x}) = \begin{Bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \tilde{u}_3 \\ \tilde{\phi} \end{Bmatrix} = \begin{Bmatrix} \widetilde{\mathbf{N}}_1 \\ \widetilde{\mathbf{N}}_2 \\ \widetilde{\mathbf{N}}_3 \\ \widetilde{\mathbf{N}}_4 \end{Bmatrix} \mathbf{d} = \widetilde{\mathbf{N}}\mathbf{d} \quad (\mathbf{x} \in \Gamma_e) \tag{20}$$

where the symbol "$\sim$" is used to specify that the field is defined on the element boundary only, $\mathbf{d} = \mathbf{d}(\mathbf{c})$ stands for the vector of the nodal displacements which are the final unknowns of the problem, $\Gamma_e$ represents the boundary of element $e$, and $\widetilde{\mathbf{N}}$ is a matrix of the corresponding shape functions which are the same as

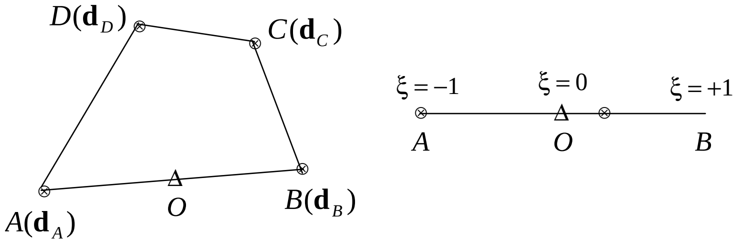


Fig. 1. A quadrilateral element generalized two-dimensional problem.

those in conventional FE formulation. For example, along the side $A$–$O$–$B$ of a particular element (see Fig. 1), a simple interpolation of the frame displacement and electric potential can be given in the form

$$\tilde{\mathbf{u}}(\mathbf{x}) = \begin{Bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \tilde{u}_3 \\ \tilde{\phi} \end{Bmatrix} = [\,\mathbf{N}_A \quad \mathbf{N}_B\,] \begin{Bmatrix} \mathbf{d}_A \\ \mathbf{d}_B \end{Bmatrix} \quad (\mathbf{x} \in \Gamma_e) \tag{21}$$

where

$$\mathbf{N}_A = \mathrm{diag}[\,N_1 \quad N_1 \quad N_1 \quad N_1\,], \qquad \mathbf{N}_B = \mathrm{diag}[\,N_2 \quad N_2 \quad N_2 \quad N_2\,], \tag{22}$$

$$\mathbf{d}_A = \{\,u_{1A} \quad u_{2A} \quad u_{3A} \quad \phi_A\,\}^{\mathrm{T}}, \qquad \mathbf{d}_B = \{\,u_{1B} \quad u_{2B} \quad u_{3B} \quad \phi_B\,\}^{\mathrm{T}} \tag{23}$$

with

$$N_1 = \frac{1-\xi}{2}, \qquad N_2 = \frac{1+\xi}{2} \tag{24}$$

Using the above definitions the generalized boundary forces and electric displacements can be derived from Eqs. (8), (9) and (13), and denoted as

$$\mathbf{T} = \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \\ D_n \end{Bmatrix} = \begin{Bmatrix} \sigma_{1j}n_j \\ \sigma_{2j}n_j \\ \sigma_{3j}n_j \\ D_j n_j \end{Bmatrix} = \begin{Bmatrix} \breve{t}_1 \\ \breve{t}_2 \\ \breve{t}_3 \\ \breve{D}_n \end{Bmatrix} + \begin{Bmatrix} \mathbf{Q}_1 \\ \mathbf{Q}_2 \\ \mathbf{Q}_3 \\ \mathbf{Q}_4 \end{Bmatrix} \mathbf{c} = \breve{\mathbf{T}} + \mathbf{Q}\mathbf{c} \tag{25}$$

where $\breve{t}_i$ and $\breve{D}_n$ are derived from $\breve{\mathbf{u}}$.

## 3. Modified variational principles

The Trefftz finite element equation for piezoelectric materials can be established by the variational approach (Qin, 2000). Since the stationary conditions of the traditional potential and complementary variational functional cannot satisfy the inter-element continuity condition which is required in the Trefftz finite element analysis, some new variational functionals need to be developed. For this purpose, we present the following four modified variational functionals suitable for Trefftz finite element analysis:

$$\Pi_m^{\sigma D} = \sum_e \Pi_{me}^{\sigma D} = \sum_e \left\{ \Pi_e^{\sigma D} - \int_{\Gamma_{De}} (\overline{D}_n - D_n)\tilde{\phi}\,\mathrm{d}s - \int_{\Gamma_{te}} (\overline{t}_i - t_i)\tilde{u}_i\,\mathrm{d}s + \int_{\Gamma_{Ie}} (D_n\tilde{\phi} + t_i\tilde{u}_i)\,\mathrm{d}s \right\} \tag{26a}$$

$$\Pi_m^{\varepsilon E} = \sum_e \Pi_{me}^{\varepsilon E}$$
$$= \sum_e \left\{ \Pi_e^{\varepsilon E} + \int_{\Gamma_{\phi e}} (\bar{\phi} - \phi)\widetilde{D}_n \, \mathrm{d}s + \int_{\Gamma_{ue}} (\bar{u}_i - u_i)\tilde{t}_i \, \mathrm{d}s - 2\int_{\Gamma_{te}} \tilde{u}_i t_i \, \mathrm{d}s - 2\int_{\Gamma_{De}} \tilde{\phi} D_n \, \mathrm{d}s - \int_{\Gamma_{Ie}} (\tilde{\phi} D_n + \tilde{u}_i t_i) \, \mathrm{d}s \right\} \tag{26b}$$

$$\Pi_m^{\varepsilon D} = \sum_e \Pi_{me}^{\varepsilon D}$$
$$= \sum_e \left\{ \Pi_e^{\varepsilon D} - \int_{\Gamma_{De}} (\overline{D}_n - D_n)\tilde{\phi} \, \mathrm{d}s + \int_{\Gamma_{ue}} (\bar{u}_i - u_i)\tilde{t}_i \, \mathrm{d}s - 2\int_{\Gamma_{te}} \tilde{u}_i t_i \, \mathrm{d}s + \int_{\Gamma_{Ie}} (D_n\tilde{\phi} - t_i\tilde{u}_i) \, \mathrm{d}s \right\} \tag{26c}$$

$$\Pi_m^{\sigma E} = \sum_e \Pi_{me}^{\sigma E}$$
$$= \sum_e \left\{ \Pi_e^{\sigma E} + \int_{\Gamma_{\phi e}} (\bar{\phi} - \phi)\widetilde{D}_n \, \mathrm{d}s - \int_{\Gamma_{te}} (\bar{t}_i - t_i)\tilde{u}_i \, \mathrm{d}s - 2\int_{\Gamma_{De}} D_n\tilde{\phi} \, \mathrm{d}s - \int_{\Gamma_{Ie}} (D_n\tilde{\phi} - t_i\tilde{u}_i) \, \mathrm{d}s \right\} \tag{26d}$$

where

$$\Pi_e^{\sigma D} = \iint_{\Omega_e} H(\boldsymbol{\sigma}, \mathbf{D}) \, \mathrm{d}\Omega + \int_{\Gamma_{ue}} t_i\bar{u}_i \, \mathrm{d}s + \int_{\Gamma_{\phi e}} D_n\bar{\phi} \, \mathrm{d}s \tag{27a}$$

$$\Pi_e^{\varepsilon E} = \iint_{\Omega_e} [H(\boldsymbol{\varepsilon}, \mathbf{E}) - \bar{b}_i u_i - \bar{q}_b\phi] \, \mathrm{d}\Omega + \int_{\Gamma_{te}} \bar{t}_i\tilde{u}_i \, \mathrm{d}s + \int_{\Gamma_{De}} \overline{D}_n\tilde{\phi} \, \mathrm{d}s, \tag{27b}$$

$$\Pi_e^{\varepsilon D} = \iint_{\Omega_e} [H(\boldsymbol{\varepsilon}, \mathbf{D}) - \bar{b}_i u_i] \, \mathrm{d}\Omega + \int_{\Gamma_{te}} \bar{t}_i\tilde{u}_i \, \mathrm{d}s + \int_{\Gamma_{\phi e}} D_n\bar{\phi} \, \mathrm{d}s \tag{27c}$$

$$\Pi_e^{\sigma E} = \iint_{\Omega_e} [H(\boldsymbol{\sigma}, \mathbf{E})] - \bar{q}_b\phi] \, \mathrm{d}\Omega + \int_{\Gamma_{ue}} t_i\bar{u}_i \, \mathrm{d}s + \int_{\Gamma_{De}} \overline{D}_n\tilde{\phi} \, \mathrm{d}s \tag{27d}$$

in which Eq. (1) is assumed to be satisfied, a priori. The terminology "modified principle" refers here to the use of a conventional functional ($\Pi_e^{xy}$ here) and some modified terms for the construction of a special variational principle to account for additional requirements such as the condition defined in Eqs. (11) and (12).

The boundary $\Gamma_e$ of a particular element consists of the following parts:

$$\Gamma_e = \Gamma_{ue} \cup \Gamma_{te} \cup \Gamma_{Ie} = \Gamma_{\phi e} \cup \Gamma_{De} \cup \Gamma_{Ie} \tag{28}$$

where

$$\Gamma_{ue} = \Gamma_u \cap \Gamma_e, \quad \Gamma_{te} = \Gamma_t \cap \Gamma_e, \quad \Gamma_{\phi e} = \Gamma_\phi \cap \Gamma_e, \quad \Gamma_{De} = \Gamma_D \cap \Gamma_e \tag{29}$$

and $\Gamma_{Ie}$ is the inter-element boundary of the element '$e$'. We now show that the stationary condition of the functional (26) leads to Eqs. (7)–(12), ($u_i = \tilde{u}_i$ on $\Gamma_t$) and ($\phi = \tilde{\phi}$ on $\Gamma_D$), and present the theorem on the existence of extremum of the functional, which ensures that an approximate solution can converge to the exact one. Taking $\Pi_m^{\sigma D}$ as an example, we have the following two statements:

(a) *Modified complementary principle*

$$\delta\Pi_m^{\sigma D} = 0 \Rightarrow (7)\text{–}(12) \quad (u_i = \tilde{u}_i \text{ on } \Gamma_t) \text{ and } (\phi = \tilde{\phi} \text{ on } \Gamma_D) \tag{30}$$

where $\delta$ stands for the variation symbol.

(b) *Theorem on the existence of extremum*
If the expression

$$\iint_\Omega \delta^2 H(\boldsymbol{\sigma}, \mathbf{D})\,\mathrm{d}\Omega + \int_{\Gamma_t} \delta t_i \delta\tilde{u}_i\,\mathrm{d}s + \int_{\Gamma_D} \delta D_n \delta\tilde{\phi}\,\mathrm{d}s + \sum_e \int_{\Gamma_{Ie}} (\delta\tilde{\phi}\delta D_n + \delta\tilde{u}_i \delta t_i)\,\mathrm{d}s \tag{31}$$

is uniformly positive (or negative) in the neighborhood of $\mathbf{u}_0$, where $\mathbf{u}_0$ is such a value that $\Pi_m^{\sigma D}(\mathbf{u}_0) = (\Pi_m^{\sigma D})_0$, and where $(\Pi_m^{\sigma D})_0$ stands for the stationary value of $\Pi_m^{\sigma D}$, we have

$$\Pi_m^{\sigma D} \geqslant (\Pi_m^{\sigma D})_0 \quad [\text{or } \Pi_m^{\sigma D} \leqslant (\Pi_m^{\sigma D})_0] \tag{32}$$

in which the relation that $\tilde{\mathbf{u}}_e = \tilde{\mathbf{u}}_f$ is identical on $\Gamma_e \cap \Gamma_f$ has been used.

**Proof.** First, we derive the stationary conditions of functional (26a). To this end, performing a variation of $\Pi_m^{\sigma D}$ and noting that Eq. (1) holds true a priori by the previous assumption, we obtain

$$\begin{aligned}\delta\Pi_m^{\sigma D} = &\int_{\Gamma_u} (\bar{u}_i - u_i)\delta t_i\,\mathrm{d}s + \int_{\Gamma_\phi} (\bar{\phi} - \phi)\delta D_n\,\mathrm{d}s - \int_{\Gamma_t} [(\bar{t}_i - t_i)\delta\tilde{u}_i - (\tilde{u}_i - u_i)\delta t_i]\,\mathrm{d}s \\ &- \int_{\Gamma_D} [(\overline{D}_n - D_n)\delta\tilde{\phi} - (\tilde{\phi} - \phi)\delta D_n]\,\mathrm{d}s + \sum_e \int_{\Gamma_{Ie}} [(\tilde{u}_i - u_i)\delta t_i + (\tilde{\phi} - \phi)\delta D_n + t_i\delta\tilde{u}_i + D_n\delta\tilde{\phi}]\,\mathrm{d}s\end{aligned} \tag{33}$$

Therefore, the Euler equations for expression (33) are Eqs. (7)–(12), ($u_i = \tilde{u}_i$ on $\Gamma_t$), and ($\phi = \tilde{\phi}$ on $\Gamma_D$), as the quantities $\delta t_i$, $\delta u_i$, $\delta\phi$, $\delta D_n$, $\delta\tilde{u}_i$ and $\delta\tilde{\phi}$ may be arbitrary. The principle (30) has thus been proved. This indicates that the stationary condition of the functional satisfies the required boundary and inter-element continuity equations and can thus be used for deriving Trefftz finite element formulation.

As for the proof of the theorem on the existence of extremum, we may complete it by way of the so-called "second variational approach" (Simpson and Spector, 1987). In doing this, performing variation of $\delta\Pi_m^{\sigma D}$ and using the constrained conditions (1), we find

$$\begin{aligned}\delta^2\Pi_m^{\sigma D} &= \iint_\Omega \delta^2 H(\boldsymbol{\sigma}, \mathbf{D})\,\mathrm{d}\Omega + \int_{\Gamma_t} \delta t_i \delta\tilde{u}_i\,\mathrm{d}s + \int_{\Gamma_D} \delta D_n \delta\tilde{\phi}\,\mathrm{d}s + \sum_e \int_{\Gamma_{Ie}} (\delta\tilde{\phi}\delta D_n + \delta\tilde{u}_i \delta t_i)\,\mathrm{d}s \\ &= \text{expression (31)}\end{aligned} \tag{34}$$

Therefore the theorem has been proved from the sufficient condition of the existence of a local extreme of a functional (Simpson and Spector, 1987). This completes the proof. The functional given in Eqs. (26b)–(26d) can be stated and proved similarly. We omit those details for the sake of conciseness. □

## 4. Element stiffness matrix

The element matrix equation can be generated by setting $\delta\Pi_{me}^{xy} = 0$. To simplify the derivation, we first transform all domain integrals in Eq. (26a) into boundary ones. In fact, by reason of the solution properties of the intra-element trial functions, the functional $\Pi_{me}^{\sigma D}$ can be simplified to

$$\begin{aligned}\Pi_{me}^{\sigma D} = &-\frac{1}{2}\int_{\Gamma_e} (t_i u_i + D_n\phi) ds - \frac{1}{2}\int_\Omega (\bar{b}_i u_i + \bar{q}_b\phi)\,\mathrm{d}\Omega - \int_{\Gamma_{De}} (\overline{D}_n - D_n)\tilde{\phi}\,\mathrm{d}s - \int_{\Gamma_{te}} (\bar{t}_i - t_i)\tilde{u}_i\,\mathrm{d}s \\ &+ \int_{\Gamma_{Ie}} (D_n\tilde{\phi} + t_i\tilde{u}_i)\,\mathrm{d}s + \int_{\Gamma_{ue}} t_i\bar{u}_i\,\mathrm{d}s + \int_{\Gamma_{\phi e}} D_n\bar{\phi}\,\mathrm{d}s\end{aligned} \tag{35}$$

Substituting the expressions given in Eqs. (13), (20), and (25) into (35) produces

$$\Pi_{me}^{\sigma D} = -\tfrac{1}{2}\mathbf{c}^{\mathrm{T}}\mathbf{H}\mathbf{c} + \mathbf{c}^{\mathrm{T}}\mathbf{S}\mathbf{d} + \mathbf{c}^{\mathrm{T}}\mathbf{r}_1 + \mathbf{d}^{\mathrm{T}}\mathbf{r}_2 + \text{terms without } \mathbf{c} \text{ or } \mathbf{d} \tag{36}$$

in which the matrices **H**, **S** and the vectors $\mathbf{r}_1$, $\mathbf{r}_2$ are defined by

$$\mathbf{H} = \int_{\Gamma_e} \mathbf{Q}^{\mathrm{T}}\mathbf{N}\,\mathrm{d}s \tag{37}$$

$$\mathbf{S} = \int_{\Gamma_{De}} \mathbf{Q}_4^{\mathrm{T}}\widetilde{\mathbf{N}}_4\,\mathrm{d}s + \int_{\Gamma_{te}} \begin{bmatrix}\mathbf{Q}_1\\ \mathbf{Q}_2\\ \mathbf{Q}_3\end{bmatrix}^{\mathrm{T}} \begin{bmatrix}\widetilde{\mathbf{N}}_1\\ \widetilde{\mathbf{N}}_2\\ \widetilde{\mathbf{N}}_3\end{bmatrix}\mathrm{d}s + \int_{\Gamma_e} \mathbf{Q}^{\mathrm{T}}\widetilde{\mathbf{N}}\,\mathrm{d}s \tag{38}$$

$$\mathbf{r}_1 = -\frac{1}{2}\int_{\Gamma_e} (\mathbf{N}^{\mathrm{T}}\breve{\mathbf{T}} + \mathbf{Q}^{\mathrm{T}}\breve{\mathbf{u}})\,\mathrm{d}s - \frac{1}{2}\int_{\Omega} \mathbf{N}^{\mathrm{T}}\bar{\mathbf{b}}\,\mathrm{d}\Omega + \int_{\Gamma_{\phi e}} \mathbf{Q}_4^{\mathrm{T}}\bar{\phi}\,\mathrm{d}s + \int_{\Gamma_{ue}} \begin{bmatrix}\mathbf{Q}_1\\ \mathbf{Q}_2\\ \mathbf{Q}_3\end{bmatrix}^{\mathrm{T}} \begin{Bmatrix}\bar{u}_1\\ \bar{u}_2\\ \bar{u}_3\end{Bmatrix}\mathrm{d}s \tag{39}$$

$$\mathbf{r}_2 = \int_{\Gamma_{De}} \widetilde{N}_4^{\mathrm{T}}(\breve{D}_n - \overline{D}_n)\,\mathrm{d}s + \int_{\Gamma_e} \widetilde{\mathbf{N}}^{\mathrm{T}}\breve{\mathbf{T}}\,\mathrm{d}s + \int_{\Gamma_{te}} \begin{bmatrix}\widetilde{\mathbf{N}}_1\\ \widetilde{\mathbf{N}}_2\\ \widetilde{\mathbf{N}}_3\end{bmatrix}^{\mathrm{T}} \left(\begin{Bmatrix}\breve{t}_1\\ \breve{t}_2\\ \breve{t}_3\end{Bmatrix} - \begin{Bmatrix}\bar{t}_1\\ \bar{t}_2\\ \bar{t}_3\end{Bmatrix}\right)\mathrm{d}s \tag{40}$$

To enforce inter-element continuity on the common element boundary, the unknown vector **c** should be expressed in terms of nodal DOF **d**. An optional relationship between **c** and **d** in the sense of variation can be obtained from

$$\frac{\partial \Pi_{me}^{\sigma D}}{\partial \mathbf{c}^{\mathrm{T}}} = -\mathbf{H}\mathbf{c} + \mathbf{S}\mathbf{d} + \mathbf{r}_1 = 0 \tag{41}$$

This leads to

$$\mathbf{c} = \mathbf{G}\mathbf{d} + \mathbf{g} \tag{42}$$

where $\mathbf{G} = \mathbf{H}^{-1}\mathbf{S}$ and $\mathbf{g} = \mathbf{H}^{-1}\mathbf{r}_1$, and then straightforwardly yields the expression of $\Pi_{me}^{\sigma D}$ only in terms of **d** and other known matrices

$$\Pi_{me}^{\sigma D} = \tfrac{1}{2}\mathbf{d}^{\mathrm{T}}\mathbf{G}^{\mathrm{T}}\mathbf{H}\mathbf{G}\mathbf{d} + \mathbf{d}^{\mathrm{T}}(\mathbf{G}^{\mathrm{T}}\mathbf{H}\mathbf{g} + \mathbf{r}_2) + \text{terms without } \mathbf{d} \tag{43}$$

Therefore, the element stiffness matrix equation can be obtained by taking the vanishing variation of the functional $\Pi_{me}^{\sigma D}$ as

$$\frac{\partial \Pi_{me}^{\sigma D}}{\partial \mathbf{d}^{\mathrm{T}}} = 0 \Rightarrow \mathbf{K}\mathbf{d} = \mathbf{P} \tag{44}$$

where $\mathbf{K} = \mathbf{G}^{\mathrm{T}}\mathbf{H}\mathbf{G}$ and $\mathbf{P} = -\mathbf{G}^{\mathrm{T}}\mathbf{H}\mathbf{g} - \mathbf{r}_2$ are, respectively, the element stiffness matrix and the equivalent nodal flow vector. The expression (44) is the elemental stiffness-matrix equation for Trefftz finite element analysis.

## 5. Numerical examples

Since the main purpose of this paper is to outline the basic principles of the TFEM in piezoelectric materials, the assessment will be limited to two simple examples. In order to allow for comparisons with other solutions appearing in references (Ding et al., 1998; Pak, 1990), the obtained results are limited to a piezoelectric prism subjected to simple tension and an anti-plane crack of length $2c$ embedded in an infinite PZT-5H medium.

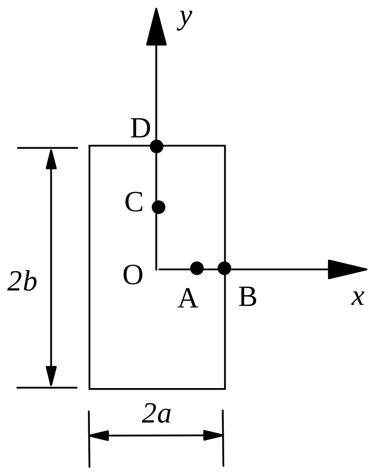


Fig. 2. Geometry of the piezoelectric prism in example.

**Example 1.** A piezoelectric prism subjected to simple tension (see Fig. 2). This example was taken from Ding et al. (1998) for a PZT-4 ceramic prism subject to a tension $P = 10\ \mathrm{N\,m^{-2}}$ in $y$-direction. The properties of the material are given as follows

$$c_{1111} = 12.6 \times 10^{10}\ \mathrm{N\,m^{-2}}, \quad c_{1122} = 7.78 \times 10^{10}\ \mathrm{N\,m^{-2}}, \quad c_{1133} = 7.43 \times 10^{10}\ \mathrm{N\,m^{-2}}$$

$$c_{3333} = 11.5 \times 10^{10}\ \mathrm{N\,m^{-2}}, \quad c_{3232} = 2.56 \times 10^{10}\ \mathrm{N\,m^{-2}}, \quad e_{131} = 12.7\ \mathrm{C\,m^{-2}}$$

$$e_{311} = -5.2\ \mathrm{C\,m^{-2}}, \quad e_{333} = 15.1\ \mathrm{C\,m^{-2}}, \quad \kappa_{11} = 730\kappa_0, \quad \kappa_{33} = 635\kappa_0$$

where $\kappa_0 = 8.854 \times 10^{-12}\ \mathrm{C^2/N\,m^2}$. The boundary conditions of the prism are

$$\sigma_{yy} = P, \quad \sigma_{xy} = D_y = 0 \quad \text{on edges } y = \pm b$$

$$\sigma_{xx} = \sigma_{xy} = D_x = 0 \quad \text{on edges } x = \pm a$$

where $a = 3$ m, $b = 10$ m. Owing to the symmetry about load, boundary conditions and geometry, only one quadrant of the prism is modeled by 10 ($x$-direction)$\times$20 ($y$-direction) elements in the TFEM analysis. Table 1 lists the displacements and electric potential at points A, B, C, and D using the present method and

Table 1

$u_1$, $u_2$, and $\phi$ of TFEM results and comparison with exact solution

| | Points | | | |
|---|---|---|---|---|
| | A(2,0) | B(3,0) | C(0,5) | D(0,10) |
| *TFEM* | | | | |
| $u_1 \times 10^{10}$ (m) | −0.9674 | −1.4510 | 0 | 0 |
| $u_2 \times 10^{9}$ (m) | 0 | 0 | 0.5009 | 1.0016 |
| $\phi$ (V) | 0 | 0 | 0.6890 | 1.3779 |
| *Exact (Pak, 1990)* | | | | |
| $u_1 \times 10^{10}$ (m) | −0.9672 | −1.4508 | 0 | 0 |
| $u_2 \times 10^{9}$ (m) | 0 | 0 | 0.5006 | 1.0011 |
| $\phi$ (V) | 0 | 0 | 0.6888 | 1.3775 |

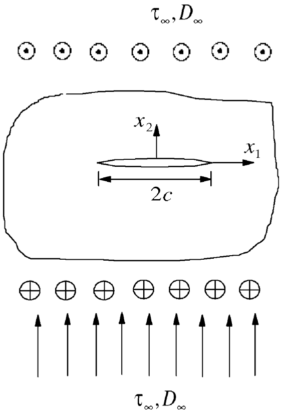


Fig. 3. Configuration of the cracked infinite piezoelectric medium.

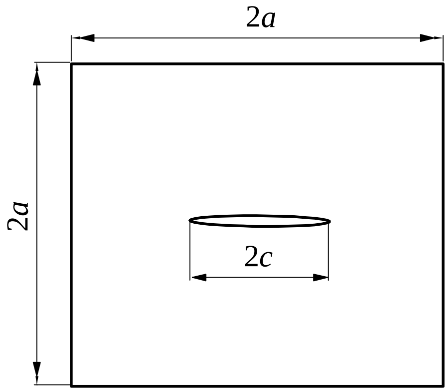


Fig. 4. Geometry of the cracked solid in finite element analysis.

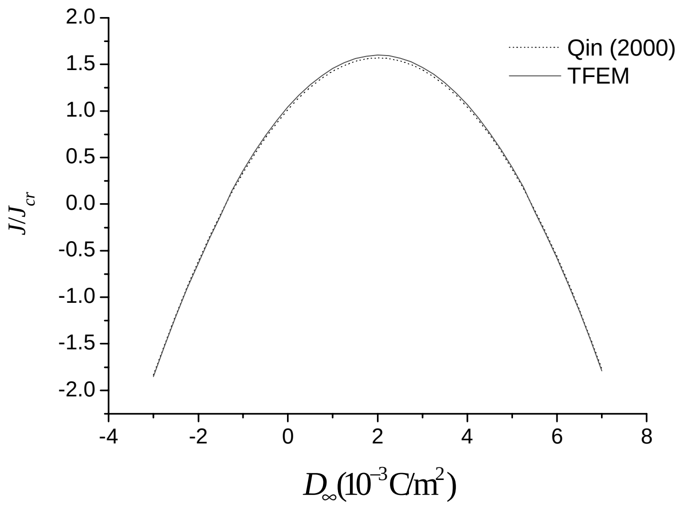


Fig. 5. Energy release rate in cracked PZT-5H plate ($a/c = 15$, $24\times24$, and $\tau_\infty = 4.2\times10^6$ N/m$^2$).

Table 2
$h$-convergence study for $J/J_{cr}$ central cracked piezoelectric plate ($a/c = 15$, $D_\infty = 2 \times 10^{-3}$ C/m$^3$, and $\tau_\infty = 4.2 \times 10^6$ N/m$^2$)

| Meshes | $J/J_{cr}$ |
|---|---|
| 8×8 | 1.5954 |
| 12×12 | 1.5908 |
| 16×16 | 1.5899 |
| 20×20 | 1.5895 |
| 24×24 | 1.5893 |

comparison is made with analytical results. It is found that the TFEM results are in good agreement with analytical ones (Ding et al., 1998).

**Example 2.** An anti-plane crack of length $2c$ embedded in an infinite PZT-5H medium. Consider an anti-plane crack of length $2c$ embedded in an infinite PZT-5H medium which is subjected to a uniform shear traction, $\sigma_{zy} = \tau_\infty$, and a uniform electric displacement, $D_y = D_\infty$ at infinity (see Fig. 3). The material properties of PZT-5H are given by Pak (1990): $c_{44} = 3.53 \times 10^{10}$ N/m$^2$, $e_{15} = 17.0$ C/m$^2$, $\kappa_{11} = 1.51 \times 10^{-8}$ C/V m, $J_{cr} = 5.0$ N/m, where $J_{cr}$ is the critical energy release rate. In the calculation, one quarter of the geometry configuration shown in Fig. 4 is used. The energy release rate for PZT-5H material with a crack of length $2c = 0.02$ m and $a/c = 15$ (it can be shown that the numerical results is nearly independent of $a/c$ if $a/c = 10$) is plotted in Fig. 5 as a function of electrical load with the mechanical load fixed such that $J = J_{cr}$ at zero electric load. The results are compared with those from Qin (2000). It is found from Fig. 5 that the energy release rate can be negative which means the crack growth may be arrested.

To study the convergent performance of the proposed formulation, numerical results for different element meshes (8×8, 12×12, 16×16, 20×20, and 24×24) are presented in Table 2 that the $h$-extension performs very nicely.

## 6. Conclusion

A family of modified variational principles of piezoelectricity is presented for Trefftz finite element analysis. It includes four variational functionals which are based on four free energy densities, each with two kinds of independent variables as basic independent variables, i.e $(\boldsymbol{\sigma}, \mathbf{D})$, $(\boldsymbol{\varepsilon}, \mathbf{E})$, $(\boldsymbol{\varepsilon}, \mathbf{D})$, and $(\boldsymbol{\sigma}, \mathbf{E})$. The proof of the stationary conditions of the variational functional and the theorem on the existence of extremum are provided in this paper. The stationary conditions are displacement and electric potential conditions on the boundary, surface traction and surface charge condition, and inter-element continuity condition. Based on the assumed intra-element and frame fields as well as the newly constructed dual variational functional, an element stiffness matrix equation is obtained which is implemented into computer programs for numerical analysis. The numerical results obtained here are in excellent agreement with the analytical ones. Besides, further extension is possible, such as the use of HFEM piezoelectric element with $p$-approach capabilities and/or some special Trefftz functions for handling local effects. The topic is under working.

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